Key

Calculus BC: Extra Practice (Series)

1.a. Write the first three nonzero terms and the general term of the Taylor series generated by $e^{\frac{x}{2}}$ at $x = 0$.

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots + \frac{u^n}{n!} + \dots$$

$$e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{(\frac{1}{2}x)^2}{2!} + \frac{(\frac{1}{2}x)^3}{3!} + \dots + \frac{(\frac{1}{2}x)^n}{n!} + \dots$$

$$e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 \cdot 3!} + \dots + \frac{x^n}{2^n \cdot n!} + \dots$$

b. Write the first three nonzero terms and the general term of a power series to represent

$$g(x) = \frac{e^x - 1}{x}.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots + \frac{x^{n-1}}{n!} + \dots$$

c. For the function g in part (b), find $g'(1)$ and use it to show that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

$$g'(x) = \frac{x \cdot e^x - (e^x - 1)}{x^2}$$

$$= \frac{x \cdot e^x - e^x + 1}{x^2}$$

$$g'(1) = \frac{1 \cdot e - e + 1}{1^2}$$

$$= 1$$

$$g'(x) = \frac{d}{dx}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} \dots \frac{x^{n-1}}{n!} + \dots\right)$$

$$g'(x) = \frac{1}{2} + \frac{2x}{3!} + \frac{3x^2}{4!} + \frac{4x^3}{5!} + \dots$$

$$g'(1) = \frac{1}{2} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots$$

$$= 1$$

2.a. Find the first four nonzero terms in the Taylor series generated by $f(x)=\sqrt{1+x}$ at $x=0$.

$f(0)=1$

$f'(x)=\frac{1}{2}(1+x)^{-1/2}$

$f''(x)=-\frac{1}{4}(1+x)^{-3/2}$

$f'''(x)=\frac{3}{8}(1+x)^{-5/2}$

$f(x)\approx 1+\frac{1}{2}x-\frac{1}{4}\frac{x^2}{2!}+\frac{3}{8}\cdot\frac{x^3}{3!}$

$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$

b. Use the results found in part (a) to find the first four nonzero terms in the Taylor series for $g(x)=\sqrt{1+x^2}$ at $x=0$.

Replace x w/ x^2

$g(x)\approx 1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6$

c. Find the first four nonzero terms in the Taylor series at $x=0$ for the function h such that $h'(x)=\sqrt{1+x^2}$ and $h(0)=5$.

derivative of

$h(x)=5+x+\frac{1}{6}x^3-\frac{1}{40}x^5$

3. What is the coefficient of x^5 in the Maclaurin series generated by $\sin 3x$?

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

replace x w/ $3x$

$$\frac{(3x)^5}{5!}$$

$$\frac{3^5}{5!} = \frac{81}{40}$$